Outline

Introduction

Sponsored search

Market with intermediaries

Introduction

Market is a system of exchange protocols

- compute the prices
- regulate the exchange

We focus on computing the prices.

An auction is a market organized by

- a seller: supply auction
- a buyer: procurement auction
Introduction

Markets in general are organized by

- universal buyers/sellers
  - merchants, traders, dealers,
  - entrepreneurs,
  - advertisers (push), solicitors (pull)
who mediate among the buyers and the sellers

- just like the universal goods
  - money
  - securities (bonds, equity, derivatives)
  - mediate among the goods

In this lecture

- Multi-item auctions
  - example: sponsored search
  - problem of incentive compatibility
    - Later: What is the value of advertising?

- Market with intermediaries
  - traders’ strategies
  - trading profits and social benefits

Outline

Introduction

Sponsored search

Sponsored search setting

Market vs auction

Generalized Second Price auction

Vickrey-Clarke-Groves Auction

Market with intermediaries

Sponsored search setting

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Sponsored search as a matching problem

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Market mechanism

- \( n \) buyers, \( n \) item
  - take \( n = 0, 1, \ldots, n - 1 \)
  - buyers valuations per item \( v = (v_i)_{i=0}^{n-1} \)
  - item prices \( p = (p_i)_{i=0}^{n-1} \)
  - matching \( \sigma_{vp} : n \to n \) assigns item \( \sigma_{vp}(i) \) to \( i \)
  - \( i \)'s utility \( u_i \in \mathbb{R} \) is
    \[
    u_i = v_{\sigma_{vp}(i)} - p_{\sigma_{vp}(i)}
    \]
Goal of the market mechanism

Maximize social welfare, i.e. buyers’ total payoff

\[ U(v, p) = \sum_{i \in \mathcal{N}} u_i \]
\[ = \sum_{i \in \mathcal{N}} v_{i\pi_i} - p_{\pi_i} \]
\[ = \sum_{i \in \mathcal{N}} v_{i\pi_i} - P \]

where \( P = \sum_{i \in \mathcal{N}} p_i \)

Markets respect preference

To maximize utility, \( \sigma : \mathcal{N} \rightarrow \mathcal{N} \) maximizes valuations

\[ v_{\sigma_i(i)} \geq v_j \]

Position auction mechanism

- \( n \) bidders, \( n \) positions
- bidders’ valuations \( v_i = w_i \cdot r \) where
  - bidders’ valuations per click \( v_i = (w_i)_n \)
  - position click-through rates \( r = (r)_n \)
- bidders bid \( b = (b)_n \)

Position auction mechanism

- \( n \) bidders, \( n \) positions
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  - bidders’ valuations per click \( v_i = (w_i)_n \)
  - position click-through rates \( r = (r)_n \)
- bidders bid \( b = (b)_n \)
- price per position \( \pi_{ij}(b) = p_i(b) \cdot r_j \) where
  - price per click \( p_i(b) = (p_i)_n \)
Position auction mechanism

- \( n \) bidders, \( n \) positions
- bidders' valuations \( v_i = w_i \cdot \eta \) where
  - bidders' valuations per click \( w = (w_i)_n \)
  - position click-through rates \( r = (r_i)_n \)
- bidders bid \( b = (b_i)_n \)
- price per position \( \pi_i(b) = p_i(b) \cdot \eta \) where
  - price per click \( p(b) = (p_i(b))_n \)
- matching \( \tau : n \times \mathbb{R}^+ \rightarrow n \) assigns item \( \tau(i, b) \) to \( i \)

Goal of the position auction mechanism

Maximize seller's revenue

\[
P(b) = \sum_{j=1}^{n} \pi_{\tau(j)}(b)
\]

\[
= \sum_{j=1}^{n} p_j(b) \cdot r_{\tau(j)}
\]

where

- all \( p_i \) grow with \( b \)
- bidder \( i \) bids \( b_i \) to maximize \( u_i(b) \).

Position auction mechanism

- \( n \) bidders, \( n \) positions
- bidders' valuations \( v_i = w_i \cdot \eta \) where
  - bidders' valuations per click \( w = (w_i)_n \)
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- price per position \( \pi_i(b) = p_i(b) \cdot \eta \) where
  - price per click \( p(b) = (p_i(b))_n \)
- matching \( \tau : n \times \mathbb{R}^+ \rightarrow n \) assigns item \( \tau(i, b) \) to \( i \)
- \( i \)'s utility \( u_i : \mathbb{R}^n \rightarrow \mathbb{R} \) is
  \[
u_i(b) = \nu_{\tau(i)}(b) - \pi_{\tau(i)}(b) = (w_i - p_i(b)) \cdot r_{\tau(i)}
\]

Assumption

- The bidders are ordered by their bids
  \[ b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \]
- The positions are ordered by click-through rates
  \[ r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_n \]

Position auctions respect preference

To maximize \( p_i(b) \) with \( u_i(b) \) always use

- \( \tau(i, b) < \tau(j, b) \implies b_i \geq b_j \), i.e.
- \( \tau(i, b) = j \) if \( b_i \) is \( j \)-th largest entry in \( b \)

Generalized Second Price Auction

- \( n \) bidders, \( n \) positions
- bidders' valuations \( v_i = w_i \cdot \eta \) where
  - bidders' valuations per click \( w = (w_i)_n \)
  - position click-through rates \( r = (r_i)_n \)
- bidders bid \( b = (b_i)_n \)
**Generalized Second Price Auction**

- $n$ bidders, $n$ positions
- bidders’ valuations $v_i = w_i \cdot r_i$ where
  - bidders’ valuations per click $w_i = (w_i)_n$
  - position click-through rates $r_i = (r_i)_n$
- bidders bid $b = (b_i)_n$
- price per click $p_i(b) = b_{i+1}$

**Does GSPA encourage truthful bidding?**

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<td>0</td>
<td>c</td>
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- with truthful bid: $u_i(7, 6, 1) = (7 - 6) \cdot 10 = 10$
- with untruthful bid: $u_i(5, 6, 1) = (7 - 1) \cdot 4 = 24$

**Position auction example**

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**Matching problem view**

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</tbody>
</table>
Idea

- How much does $x$ subtract from social welfare?

slots | advertisers | valuations
---|---|---
$\times$ | $\times$ | 30, 15, 6
$b$ | $\times$ | 20, 10, 4
$c$ | $\times$ | $10^5\times$

If $x$ weren't there, $y$ would do better by 20–10–10, and $z$ would do better by 5–0–3, for a total harm of 13.

Traders

Idea: Vickrey, Clarke, Groves

- Each bidder should pay the cost that their bid incurs on social welfare
  - i.e., the sum of the losses that they cause to other bidders

Vickrey-Clarke-Groves Auction

Notation

- $B$ — set of bidders
- $S$ — set of sellers (items)
- $v = (v_b)_{b \in B}$ — bidders’ valuations
- $V_S^B$ — maximal total valuation

Remark

- If $\#B < \#S$, then add $\#S - \#B$ bidders with all valuations 0
- If $\#B > \#S$, then add $\#B - \#S$ sellers valued 0 by all.
Vickrey-Clarke-Groves Auction

- n bidders, n positions
- bidders’ valuations $v_{ij} = w_i \cdot r_j$ where
  - bidders’ valuations per click $w_i$, position click-through rates $r_j$.
- bidders bid $b = (b_i)_n$
- price per item $p_{ij}(b) = V_{B_i} - V_{B_j}$
- i's utility $u_i : \mathbb{R}^n \to \mathbb{R}$ is
  $u_i(b) = v_{ii} - \pi_i(b)$

Theorem

The VCG auction is incentive compatible: truthful bidding is the unique Nash equilibrium for all players.

Corollary

The VCG auction maximizes social welfare, i.e. the total utility of bidders.

Problem

Homework

For the sponsored search market

<table>
<thead>
<tr>
<th>item</th>
<th>clicks</th>
<th>views</th>
<th>clicks per view</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3</td>
<td>1.67</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>3</td>
<td>2.33</td>
</tr>
<tr>
<td>5</td>
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<td>4</td>
<td>0.5</td>
</tr>
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compute seller’s revenue (i.e. the total of the prices charged for all items) if the positions are auctioned by a GSP auction and by a VCG auction

Show that neither of these mechanisms maximizes seller’s revenue.
Billion $ problem

Design an auction mechanism that maximizes seller's revenue.

Outline

Introduction

Sponsored search

Market with intermediaries

Toy market

- There is just one type of goods.
- Every buyer needs to buy one item.
- Every seller needs to sell one item.

Goal of the market

Find a bijection \( \sigma : \mathcal{B} \rightarrow \mathcal{S} \) that maximizes social benefit

\[
SB_{\sigma} = \sum_{i=1}^{n} v_i - w_{\sigma(i)}
\]
Market with intermediaries

- Just like the goods are compared through universal goods
  - money, securities
- the buyers' and the sellers' are connected through universal buyers/sellers
  - merchants, traders, advertisers

The intermediaries mediate the flows

- merchants buy, move and sell goods
- traders buy and sell goods without moving them
- advertisers and solicitors move information

Setting

- buyers $\mathcal{B} = \{B_1, B_2, B_3\}$
  - $B_i$'s reserve price (valuation) is $v_i$
- sellers $\mathcal{S} = \{S_1, S_2, S_3\}$
  - $S_j$'s reserve price (valuation) is $w_j$
- traders $\mathcal{T} = \{T_1, \ldots, T_m\}$
  - ask relation $\sim_A : \mathcal{T} \times \mathcal{B}$
    - $T_i$'s buyers $B_k = \{B \in \mathcal{B} | T_i \sim_A B\}$
  - bid relation $\sim_B : \mathcal{T} \times \mathcal{S}$
    - $T_i$'s sellers $S_j = \{S \in \mathcal{S} | S \sim_B T_i\}$

Market with intermediaries as a game

players: $T_1, \ldots, T_m$
moves: for the trader $T_k$ the set of moves is

$$P_k = P_B \times P_A,$$

where

$$P_B = \mathbb{R}^p$$

and

$$P_A = \mathbb{R}^q$$

with $p = |\mathcal{S}_k|$ and $q = |\mathcal{B}_k|$

$B_k = (b_{k1}, b_{k2}, \ldots, b_{kp}) \in P_B$ are $T_k$'s bid prices for all $S_j \in \mathcal{S}_k$

$A_k = (a_{k1}, a_{k2}, \ldots, a_{kq}) \in P_A$ are $T_k$'s ask prices for all $B_j \in \mathcal{B}_k$
Market with intermediaries as a game

### Play
- Each $T_i$ announces its bid and ask prices $\rho_k = (a_k, b_k)$
- Each $S_j$ agrees to sell to a $T_i$ with a maximal $b_j$
- Each $B_k$ agrees to buy from a $T_i$ with a minimal $a_k$

Each $T_i$ thus forms the sets of:
- Suppliers $\text{MS}_k = \{S_j \in S_j \mid \forall b_i \leq b_j\}$
- Customers $\text{MB}_k = \{B_k \in B_k \mid \forall a_i \geq a_k\}$

### Distribution of social benefit

If the bijection $\sigma : S \rightarrow T$ that maximizes social benefit

$$SB_\sigma = \frac{\sum_{i=1}^{n} v_i - w_i}{m}$$

is found through the traders $\kappa : S \rightarrow T$, then the benefit is distributed

$$SB_\kappa = \frac{\sum_{i=1}^{n} (v_i - a_{\sigma(i)}) + (a_{\sigma(i)} - b_{\kappa(i)}) + (b_{\kappa(i)} - w_i)}{m}$$

where
- $UB$ is the utility of the buyer
- $UT$ is the utility of the trader
- $US$ is the utility of the seller

### Distribution of social benefit

If the bijection $\sigma : S \rightarrow T$ that maximizes social benefit

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where
- $UB$ is the utility of the buyer
- $UT$ is the utility of the trader
- $US$ is the utility of the seller

The traders maximize $UT$.

### Implicit perfect competition

**Dusko Pavlovic**

**II-5.**

**Introduction**

**Market with intermediaries as a game**

Trader $T_i$’s utility
- If $\#\text{MB}_k \leq \#\text{MS}_k$ (sufficient supplies) then
  $$u_k(\beta) = \frac{\sum_{b_i \in \text{MB}_k} a_i - \sum_{b_i \in \text{MS}_k} b_i}{n}$$
- If $\#\text{MB}_k > \#\text{MS}_k$ (insufficient supplies) then
  $$u_k(\beta) = \frac{\sum_{b_i \in \text{MB}_k} a_i - \sum_{b_i \in \text{MS}_k} b_i - \sum_{a_i \notin \text{MS}_k} a_i}{n}$$

where $\text{MS}_k - \text{MB}_k$ and $\text{MB}_k - \text{MS}_k$ are sets of buyers.

**Distribution of social benefit**

- But how do the traders achieve their payoffs?
- What are the equilibria in the trading game?
Indifference principle

At equilibrium

- All bid prices offered to a seller must be equal
- The seller will accept the bid from the trader who has access to the highest paying buyers
  - because that trader can increase the bid by $\epsilon$

- All ask prices offered to a buyer must be equal
- The buyer will accept the offer from the trader who has access to the lowest charging sellers
  - because that trader can undercut the offer by $\epsilon$

Ripple effects

0 $\leq$ $x$ $\leq$ 2

1 $\leq$ $y$ $\leq$ 2  1 $\leq$ $z$ $\leq$ 3